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WIDE-BAND RANDOM VIBRATION IN MEMBERS OF COMPLEX STRUCTURES

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Abstract—Bolotin's method of integral estimates, Statistical Energy Analysis and the theory of high-frequency vibration are combined in order to predict the vibrations of structural members in complex systems. The concept allows us to consider any driven component separately, where the presence of other structural members and coupling are taken into account by means of a frequency drift and an effective damping factor. The utility of the concept is demonstrated by solving two practical problems: (i) vibration of a solid-propellant engine caused by its vibrating combustion; (ii) vibrations of a thin-walled cover of an engine head. The approach enables one to indicate a frequency domain wherein a localization of vibration within structural members occurs.

1. INTRODUCTION

In engineering practice one is often faced with the problem of a predicting the vibration behaviour of complex structures. A detailed description of the vibration of a complex structure is made difficult, first, by the complexity of the structure's shape, then by the assemblage of individual substructures and, finally, by the presence of various secondary systems attached to the primary structure. Even if one could obtain an "exact" solution the very interpretation of this result is difficult. The vibration field of modern complex structures under broad-band excitation is a complicated function of both time and spatial coordinates, since so many modes are excited in the structure.

Under these circumstances it seems reasonable to use integral descriptions. In fact, all conventional methods of structural mechanics (substructure synthesis method, FEM, BEM, etc.) are based on an integral representation (Meirovitch, 1980). They work unconditionally well in the frequency domain of the global resonances of complex structures. However, this domain is not large. According to engineering experience only the first, the second and rarely the third global resonances occur in actual complex structures. This region is a frequency domain of very low eigenfrequencies where the rigidity of the complex structure is sufficient to propagate the influence of boundary conditions throughout the whole structure. As methods of structural mechanics advance, it seems evident that the forthcoming structures will possess even lower global rigidity.

Outside of the region of these global resonances, however, local vibrations exist within each substructure or groups of substructures, thus different approaches are needed to deal with those vibrations. First, the Bolotin method of integral estimates (Bolotin, 1984) is worth mentioning. The central idea of the method implies the consideration of a substructure as an absolutely isolated one. This method was applied to estimate a certain mean value of vibration of thin-walled elements under broad-band excitation. Its application results in obtaining rather simple formulae in closed form. Instead of eigenfrequencies and normal modes one may use their asymptotic expressions in the high-frequency domain. The main shortcoming of the Bolotin approach is the absolute isolation of substructures

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considered. Contrary to this, Statistical Energy Analysis (SEA), being a simple, diffusive, transport theory, is aimed mainly at studying the mechanical properties of the coupling between the substructures (Lyon, 1975). A complex structure is viewed as a set of coupled substructures and dynamic analysis deals only with the spatially averaged vibration within each substructure. The advantages and shortcomings of SEA are known, for details one can see the comprehensive reviews by Fahy (1974, 1994) and Hodges and Woodhouse (1986).

Certain integral methods have been offered by Belyaev and Palmov (1986) to predict the field of high-frequency vibration of complex structures. The main distinctions between low- and high-frequency vibrations as well as some essential features of high-frequency vibration are described in the paper by Belyaev (1991). As shown in these papers a relatively simple boundary-value problem for high-frequency vibration is obtained. The properties of an actual structure are reflected in this integral theory in the form of certain averaged rigidity and averaged mass characteristics, and generalized spectra. This enables us to avoid dipping into too many details of the structure. This theory can be used to obtain some generalized characteristics of the vibration field. This level of description can be considered sufficient for many cases. Nonetheless, it is often necessary to know the vibration of a particular member of the structure. This cannot be managed with the integral methods alone, since the member itself is not represented in the dynamical model. Thus, to analyse the behaviour of a particular member one has to take into account both its individual mechanical characteristics and the nature of its interaction with other structural members. As has already been mentioned, trying to cover the entire structure makes this task completely hopeless due to the great computational difficulties. Thus, a more realistic concept lies in the precise consideration of this member alone, with the rest of the structure being described integrally.

This paper is an attempt to combine these three aforementioned approaches.

2. LOCAL PRINCIPLE IN STRUCTURAL DYNAMICS

A combination of integral and precise descriptions, namely, the local principle in structural dynamics, was proposed by Belyaev and Palmov (1984); see also Belyaev (1990). A brief description of this concept is given below. All up-to-date complex structures are actually assemblages of substructures attached to one another or to the primary structure. As already mentioned, the field of displacement of a complex structure is a very complicated function of spatial coordinates for each frequency. Nevertheless, it turns out to be possible to select from the field of displacement a function $\mathbf{u}(\mathbf{r}, \omega)$, the displacement of a framework (or displacement of a carrier structure). This function coincides with the actual displacement on all substructural boundaries, but it has the property of extreme spatial smoothness within the whole structure. The assumption that $\mathbf{u}$ is a smooth function compared with normal modes and with the mechanical characteristics of essentially heterogeneous structures results in the following boundary-value problem for $\mathbf{u}(\mathbf{r}, \omega)$ [for details see Belyaev and Palmov (1986) and Belyaev (1991)]:

$$\text{in volume } V: \quad \nabla \cdot (\langle \mathbf{C} \rangle : \nabla \mathbf{u}) + \omega^2 A(\omega)\mathbf{u} + \mathbf{f} = \mathbf{0} \tag{1}$$

$$\text{on boundary } B: \quad \mathbf{N} \cdot (\langle \mathbf{C} \rangle : \nabla \mathbf{u}) = \mathbf{F}, \tag{2}$$

where $\langle \mathbf{C} \rangle$ is an averaged tensor of elastic moduli, $\mathbf{f}$ and $\mathbf{F}$ are the spectra of external volume and surface loads, ∇ is the Hamiltonian operator, the symbols $\cdot$ and $:$ denote the scalar and double scalar product, and $\mathbf{N}$ is the unit vector of external normal. A close consideration is required for the dynamical inertia $A(\omega)$ [e.g. Belyaev (1991)]:

$$A(\omega) = \langle \rho \rangle \left[1 + \tfrac{1}{3} \langle \rho \rangle \omega^2 V_n \sum_{k=1}^{L} \frac{\langle \mathbf{u}_{nk} \rangle \cdot \langle \mathbf{u}_{nk} \rangle}{-\omega^2 + 2i\psi_{nk}\Omega_{nk}\omega + \Omega_{nk}^2} \right], \tag{3}$$

where $\langle \rho \rangle$ is an averaged density, $\langle \mathbf{u}_{nk} \rangle$ is an average displacement of the centre of mass of the substructure V_n when it moves according to the mode k, Ω_{nk} is the kth eigenfrequency of substructure V_n, and ψ_{nk} is the critical damping ratio. Due to eqn (3), $A(\omega)$ is a superposition of single-degree-of-freedom resonance curves. The width of each resonance curve is $2\psi_{nk}\Omega_{nk}$ at the "3 dB below" or "half-power" level. If the resonant width is large compared to the eigenfrequency separation $\Delta\Omega_{nk}$ (high modal overlap), i.e.

$$\Delta\Omega_{nk} = \Omega_{nk+1} - \Omega_{nk} \leqslant \psi_{nk}\Omega_{nk} + \psi_{nk+1}\Omega_{nk+1} \quad \text{or} \quad \frac{\Delta\Omega_{nk}}{\Omega_{nk}} \leqslant 2\psi_{nk}, \tag{4}$$

the resonance curves in eqn (3) merge to form a smooth frequency function. In this case the sum in eqn (3) can be replaced by the integral over the high-frequency domain

$$A(\omega) = \langle \rho \rangle \left[1 + \omega^2 \int_{\Theta}^{+\infty} \frac{\Phi(\Omega)\,\mathrm{d}\Omega}{-\omega^2 + 2i\psi\Omega\omega + \Omega^2} \right], \tag{5}$$

where a smooth function of the eigenfrequencies distribution $\Phi(\Omega)$ is introduced; Θ is the frequency obtained from the equality condition in eqn (4). The relative density of the spectrum of the eigenfrequencies of any mechanical system is known to increase with the growth of the ordinal number of the eigenfrequency (Bolotin, 1984). Hence, for the frequency range $\Omega > \Theta$, the condition (4) and the replacement of the sum by the integral is valid. The frequency domain $\Theta < \omega < \infty$ is referred to as the high-frequency region. Obviously Θ is specific for each mechanical structure and it depends primarily on the relative density of eigenfrequencies and damping value [cf. eqn (4)].

The following representation was recommended by Belyaev and Palmov (1986) and by Belyaev (1991):

$$A(\omega) = \langle \rho \rangle [1 - i\kappa(\omega)]^2, \quad \kappa(\omega) = \omega^3 \int_{\Theta}^{\infty} \frac{\psi\Omega\Phi(\Omega)\,\mathrm{d}\Omega}{(\Omega^2 - \omega^2)^2 + 4\psi^2\Omega^2\omega^2}, \tag{6}$$

where $\kappa(\omega)$ is a non-dimensional frequency-dependent parameter, which is responsible for the absorption of high-frequency vibration by the structure. In the case of light damping $\psi \ll 1$ and local smooth function $\Phi(\Omega)$, one can carry out an approximate calculation of the integral in eqn (6). Integrals of this kind are encountered in calculating the dispersion of steady-state random vibration fields in systems with small damping [e.g. Bolotin (1984)]. Using this formalism we obtain

$$\kappa(\omega) = \frac{\pi}{2}\omega\Phi(\omega). \tag{7}$$

From the latter equation we can see that $\kappa(\omega)$ and consequently the value of the absorption of high-frequency vibration does not depend on the damping ψ at all. This absorption is determined first of all by the distribution function $\Phi(\omega)$. This means that the secondary systems act as a set of dynamical absorbers with respect to the primary structure. Since the resonance curves corresponding to the internal degrees of freedom merge, a considerable spatial absorption of vibration for the whole high-frequency region is observed.

This integral description can be applied for all substructures except the substructure of interest. It allows us to formulate the local principle (Belyaev and Palmov, 1984; Belyaev, 1990): for each structure characterized by a certain relative density of the eigenfrequency spectrum and damping there is a critical frequency Θ. If it is exceeded ($\omega > \Theta$), then any substructure of interest can be described precisely, while the others are described integrally by means of the boundary-value problem, eqns (1) and (2). In other words, the vibration of an individual substructure in the structure depends mainly upon (i) the mechanical characteristics of this substructure itself, (ii) its particular composition within the structure, and (iii) certain generalized mechanical properties of the structure. It is worth mentioning that the vibration of any individual substructure does not depend upon details of remote substructures.

Some examples of application of the locality principle are given in the paper by Belyaev (1990), where it was shown that consideration of a driven substructure as an absolutely isolated one is not correct in the general case. On the other hand, ignoring the coupling properties may lead to a dubious result such as an absence of local resonances in weakly coupled structures.

3. THE BOUNDARY-VALUE PROBLEM

The field of high-frequency vibration of actual complex structures is three-dimensional. Nevertheless, as shown by Pervozvanskii (1986), high-frequency vibrations propagate along wave-guides as uniaxial waves. This means that for extended complex structures one can apply the theory of high-frequency vibration in one dimension. In Fig. 1 a complex structure is depicted as a rod $0 \leqslant x \leqslant L$ described integrally. The substructure of interest is represented as a rod $l \leqslant x \leqslant r$.

The coupling is commonly considered to be conservative (Lyon, 1975; Fahy, 1974; Hodges and Woodhouse, 1986). Gaskets and contacts, as the two most typical cases of coupling in engineering mechanics, will be under our consideration. Contact of two surfaces is inherently imperfect, and the actual contact takes place at a limited number of spots. According to experimental data of Levina and Reshetov (1971), an actual surface of the contact is smaller than the nominal one by a factor of 2×10^5 under a pressure of 0.15×10^5 N m^{-2}, and by a factor of 130 under a pressure of 15×10^5 N m^{-2}. This results in a very low local contact rigidity and in massless coupling. The same conclusions are valid for gaskets. In all cases it allows us to model the coupling by a spring of rigidity K.

In order to obtain the boundary-value problem let us consider the subsystem $l \leqslant x \leqslant r$. Its dynamics are governed by the equation

$$(C_2 u_2')' - \mu_2 \ddot{u}_2 + f = 0; \quad \dot{} = \frac{\partial}{\partial t}, \quad ' = \frac{\partial}{\partial x}, \tag{8}$$

where $C_2(x)$ and $\mu_2(x)$ are the longitudinal rigidity and mass density per unit length respectively, $u_2(x, t)$ is the absolute displacement, and $f(x, t)$ is the external distributed force.

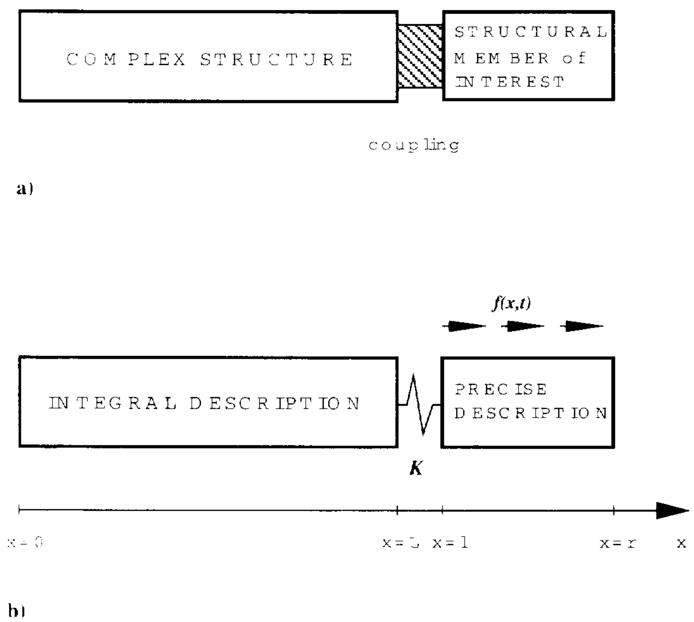

Fig. 1. A structural member (a) and its mechanical model due to the local principle (b).

Let $v_n(x)$, $n = 1, 2, \ldots, \infty$ be the normal modes of the free rod $l \leqslant x \leqslant r$, which are supposed to be normalized, i.e.

$$\int_l^r \mu_2(x) v_k(x) v_n(x)\,\mathrm{d}x = \delta_{kn}; \quad \int_l^r C_2(x) v_k'(x) v_n'(x)\,\mathrm{d}x = \Omega_n^2 \delta_{kn}, \tag{9}$$

where Ω_n is the nth eigenfrequency. The set of normal modes v_n' is known to be complete within this rod. In order to obtain the equation for the generalized coordinate $q_n(t)$

$$q_n(t) = \int_l^r \mu_2(x) v_n(x) u_2(x, t)\,\mathrm{d}x, \tag{10}$$

we multiply eqn (8) by $v_n(x)$ and integrate along the length of the rod. After integration by parts we have

$$\ddot{q}_n + \Omega_n^2 q_n = \int_l^r f v_n\,\mathrm{d}x + [(C_2 u_2')v_n - u_2(C_2 v_n')]\Big|_{x=l}^{x=r}. \tag{11}$$

Taking into account the properties of the normal modes $v_n'(l) = v_n'(r) = 0$ and the boundary condition $u_2'(r) = 0$ gives

$$\ddot{q}_n + \Omega_n^2 q_n = \int_l^r f(x, t) v_n(x)\,\mathrm{d}x - [C_2 u_2'(l, t)] v_n(l). \tag{12}$$

Let us assume that the external load, generalized coordinates etc. can be represented by their spectral representations

$$f(x, t) = \int_{-\infty}^{\infty} f(x, \omega)\,\mathrm{e}^{i\omega t}\,\mathrm{d}\omega, \quad q_n(t) = \int_{-\infty}^{\infty} q_n(\omega)\,\mathrm{e}^{i\omega t}\,\mathrm{d}\omega \tag{13}$$

and so on, where from now on the same designation for the spectra will be retained. Substituting eqn (13) into (12) gives us the following expression for the spectrum of the generalized coordinates:

$$q_n(\omega) = \frac{1}{-\omega^2 + \Omega_n^2}\left\{\int_l^r f(x, \omega) v_n(x)\,\mathrm{d}x - [C_2 u_2'(l, \omega)] v_n(l)\right\}. \tag{14}$$

The differential equation for integral description of the part $0 \leqslant x \leqslant L$ is given as

$$0 < x < L \quad C_1 \frac{\mathrm{d}^2 u_1}{\mathrm{d}^2 x} + M_1 \omega^2 (1 - i\kappa)^2 u_1 = 0, \tag{15}$$

where M_1 and C_1 are the averaged mass per unit length and the averaged longitudinal rigidity of the structure, respectively, $\kappa(\omega)$ is a dimensionless frequency-dependent parameter responsible for the absorption of high-frequency vibration by the structure, and u_1 is the absolute displacement of the framework. Balancing the coupling forces yields

$$C_1 u_1'(L) = K[u_2(l) - u_1(L)] = C_2 u_2'(l). \tag{16}$$

As shown by Belyaev (1991), the field of high-frequency vibration is not actually sensitive to boundary conditions at unloaded remote boundaries of extended structure. This allows us to set e.g. the following simple boundary condition:

$$x = 0 \quad u_1 = 0. \tag{17}$$

4. EFFECTIVE MECHANICAL CHARACTERISTICS OF VIBRATING COMPONENTS

The solution of eqns (15) and (17) is

$$u_1(x,\omega) = A\sin\frac{\omega(1-i\kappa)x}{a_1}, \tag{18}$$

where $a_1 = \sqrt{C_1/M}$ is the group velocity. With the help of eqn (16) the result (14) can be rewritten as

$$q_n(\omega) = \frac{1}{-\omega^2+\Omega_n^2}\left\{\int_l^r f(x,\omega)v_n(x)\,\mathrm{d}x - K[u_2(l,\omega)-u_1(L,\omega)]v_n(l)\right\} \tag{19}$$

and, correspondingly,

$$u_2(l,\omega) = \sum_{n=1}^{\prime} v_n(l)q_n(\omega) = \sum_{n=1}^{\prime} \frac{v_n(l)\int_l^r f(x,\omega)v_n(x)\,\mathrm{d}x - K[u_2(l,\omega)-u_1(L,\omega)]v_n^2(l)}{-\omega^2+\Omega_n^2}. \tag{20}$$

Equation (16) helps us to express eqn (18) in such a way that

$$u_1(L,\omega) = C_1u_1'(L)\frac{\tan\dfrac{\omega(1-i\kappa)L}{a_1}}{C_1\dfrac{\omega(1-i\kappa)}{a_1}} = K[u_2(l,\omega)-u_1(L,\omega)]\frac{\tan\dfrac{\omega(1-i\kappa)L}{a_1}}{C_1\dfrac{\omega(1-i\kappa)}{a_1}}. \tag{21}$$

Subtracting eqn (21) from (20) allows us to obtain the following expression for $u_2(l,\omega)-u_1(L,\omega)$:

$$u_2(l,\omega)-u_1(L,\omega) = \frac{\displaystyle\sum_{n=1}^{\prime}\frac{v_n(l)}{-\omega^2+\Omega_n^2}\int_l^r fv_n\,\mathrm{d}x}{1+K\left[\displaystyle\sum_{n=1}^{\prime}\frac{v_n^2(l)}{-\omega^2+\Omega_n^2}+\frac{a_1}{C_1\omega(1-i\kappa)}\tan\frac{\omega(1-i\kappa)L}{a_1}\right]}. \tag{22}$$

The generalized coordinate, eqn (19), is now as follows:

$$q_n(\omega) = \frac{1}{-\omega^2+\Omega_n^2} \times\left\{\int_l^r f(x,\omega)v_n(x)\,\mathrm{d}x - \frac{Kv_n(l)\displaystyle\sum_{j=1}^{\prime}\frac{v_j(l)}{-\omega^2+\Omega_j^2}\int_l^r fv_j\,\mathrm{d}x}{1+K\left[\displaystyle\sum_{j=1}^{\prime}\frac{v_j^2(l)}{-\omega^2+\Omega_j^2}+\frac{a_1}{C_1\omega(1-i\kappa)}\tan\frac{\omega(1-i\kappa)L}{a_1}\right]}\right\}. \tag{23}$$

Simple transformations give

$$q_n(\omega) = \frac{\left\{1+K\left[\sum_{\substack{j=1\\j\neq n}}^{\infty}\frac{v_j^2(l)}{-\omega^2+\Omega_j^2}+\frac{\tan\omega(1-i\kappa)L/a_1}{C_1\omega(1-i\kappa)/a_1}\right]\right\}\int_l^r fv_n\,\mathrm{d}x - Kv_n(l)\sum_{\substack{j=1\\j\neq n}}^{\infty}\frac{v_j(l)}{-\omega^2+\Omega_j^2}\int_l^r fv_j\,\mathrm{d}x}{(-\omega^2+\Omega_n^2)\left\{1+K\left[\sum_{j=1}^{\infty}\frac{v_j^2(l)}{-\omega^2+\Omega_j^2}+\frac{\tan\omega(1-i\kappa)L/a_1}{C_1\omega(1-i\kappa)/a_1}\right]\right\}}. \tag{24}$$

In order to produce an integral description of the random vibration in the structural member let us make use of the method of integral estimates (Bolotin, 1984). A mean value of vibration of a component could be evaluated using asymptotic expressions for eigenfrequencies and normal modes. A generalisation of this method given by Palmov (1979) does not require any information about normal modes at all. Following Palmov (1979), let us consider the case $\mu_2(x) = \mu = \text{const}$. In this case an averaged value of the square of the displacement is given as

$$\langle u_2^2\rangle = \frac{1}{r-l}\int_l^r E\{u_2^2(x,t)\}\,\mathrm{d}x = \frac{1}{\mu(r-l)}\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}E\{q_jq_n\}\int_l^r \mu v_n v_j\,\mathrm{d}x = \frac{1}{\mu(r-l)}\sum_{n=1}^{\infty}E\{q_n^2(t)\}, \tag{25}$$

where an ensemble average $E\{\ \}$ covers different realization of the applied random loading.

Let us suppose that external force f is a stationary delta-correlated spatial white noise. Mathematically this can be expressed in the following form:

$$E\{f(x,\omega)f^*(x',\omega')\} = S_f(\omega)\delta(\omega-\omega')\delta(x-x'), \tag{26}$$

where $S_f(\omega)$ is the spectral density of the external force and * denotes the complex conjugate of a quantity. Provided that $\mu(x) = \text{const}$ and the normal modes are normalised, cf. eqn (9), the substitution of eqn (24) into (25) gives

$$\langle u_2^2\rangle = \frac{1}{\mu(r-l)}\sum_{n=1}^{\infty}E\{q_n^2(t)\}$$

$$= \frac{1}{\mu^2(r-l)}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\frac{S_f(\omega)\,\mathrm{d}\omega}{|-\omega^2+i\xi_n\omega+\Omega_n^2|^2\left|1+K\left[\sum_{j=1}^{\infty}\frac{v_j^2(l)}{-\omega^2+i\xi_j\omega+\Omega_j^2}+\frac{\tan\omega(1-i\kappa)L/a_1}{C_1\omega(1-i\kappa)/a_1}\right]\right|^2}$$

$$\times\left\{\left|1+K\left[\sum_{\substack{j=1\\j\neq n}}^{\infty}\frac{v_j^2(l)}{-\omega^2+i\xi_j\omega+\Omega_j^2}+\frac{\tan\omega(1-i\kappa)L/a_1}{C_1\omega(1-i\kappa)/a_1}\right]\right|^2+K^2v_n^2(l)\sum_{\substack{j=1\\j\neq n}}^{\infty}\frac{v_j^2(l)}{|-\omega^2+i\xi_j\omega+\Omega_j^2|^2}\right\}, \tag{27}$$

where internal damping is introduced via the viscous damping coefficient ξ_n.

In order to simplify this formula one should observe that $\langle u_2^2\rangle$ is formed by an infinite number of resonant curves corresponding to the vibration of a single-degree-of-freedom system. If the modal overlap is high the aforementioned formalism of the theory of high-frequency vibration can be applied. Our interest lies in the case when the adjacent resonant curves of the component under consideration are rather distant from each other (i.e. low modal overlap). In this case considerable simplification is possible. In the vicinity of an eigenfrequency Ω_n the resonant term prevails and there are no large components in the sum

$$\sum_{\substack{j=1 \\ j \neq n}}^{r} \frac{v_j^2(l)}{-\omega^2 + i\xi_j\omega + \Omega_j^2}. \tag{28}$$

This allows us to neglect the sums of this type appearing in eqn (27) to obtain

$$\langle u_\xi^2 \rangle = \frac{1}{\mu^2(r-l)} \sum_{n=1}^{r} \int \frac{S_f(\omega)\,\mathrm{d}\omega}{\left| -\omega^2 + i\xi_n\omega + \Omega_n^2 + \dfrac{Kv_n^2(l)}{1 + K\dfrac{\tan\left[\omega(1-i\kappa)L/a_1\right]}{C_1\omega(1-i\kappa)/a_1}} \right|^2}. \tag{29}$$

The latter term in the denominator represents the effect of the backward influence of the whole structure on the substructure vibrations. For extended complex structures at high frequencies the following asymptotic estimation is valid:

$$\tan\left[\omega(1-i\kappa)L/a_1\right] = -i\frac{\exp i\left[\omega(1-i\kappa)L/a_1\right] - \exp -i\left[\omega(1-i\kappa)L/a_1\right]}{\exp i\left[\omega(1-i\kappa)L/a_1\right] + \exp -i\left[\omega(1-i\kappa)L/a_1\right]} = -i,$$

which is confirmed by the observation that the impedance of a mechanical system is equal to the impedance of an infinite system provided that the impedance is measured in a sufficiently broad frequency band (Remington and Manning, 1975). Now eqn (29) can be rewritten as

$$\langle u_\xi^2 \rangle = \frac{1}{\mu^2(r-l)} \sum_{n=1}^{r} \int \frac{S_f(\omega)\,\mathrm{d}\omega}{\left| -\omega^2 + i\hat{\xi}_n\omega + \hat{\Omega}_n^2 \right|^2}, \tag{30}$$

where an effective eigenfrequency $\hat{\Omega}_n$ and an effective damping $\hat{\xi}_n$ are introduced

$$\hat{\Omega}_n^2 = \Omega_n^2 + Kv_n^2(l)\left[1 + \frac{\kappa a_1 K}{C_1\Omega_n(1+\kappa^2)}\right]\left\{\left[1 + \frac{\kappa a_1 K}{C_1\Omega_n(1+\kappa^2)}\right]^2 + \left[\frac{a_1 K}{C_1\Omega_n(1+\kappa^2)}\right]^2\right\}^{-1} \tag{31}$$

$$\hat{\xi}_n = \xi_n + \frac{a_1 K^2 v_n^2(l)}{C_1\Omega_n^2(1+\kappa^2)}\left\{\left[1 + \frac{\kappa a_1 K}{C_1\Omega_n(1+\kappa^2)}\right]^2 + \left[\frac{a_1 K}{C_1\Omega_n(1+\kappa^2)}\right]^2\right\}^{-1}. \tag{32}$$

Equation (30) is a standard one for wide-band vibration of a rod. Hence, instead of consideration of the whole structure as two coupled mechanical systems with distinctive characteristics and properties, it becomes convenient to consider only the substructure of interest. The influence of both the rest structure and the coupling is taken into account by the effective eigenfrequencies $\hat{\Omega}_n$ and the effective damping factor $\hat{\xi}_n$.

5. APPLICATIONS OF THE CONCEPT

The power of the concept is demonstrated by solving two practical problems.

5.1. *Vibrating combustion of a solid propellant engine*

High-amplitude pressure oscillations have been observed in various stick propellants, and they influence significantly the vibration field of a rocket. Such a phenomenon is usually referred to as vibrating combustion of a solid propellant. In the past, vibrating combustion, which is in fact an erosive burning, was studied extensively [e.g. Kuo and Summerfield (1984); Hsieh and Kuo (1990)]. As shown in these papers, such a combustion may be considered as a wide-band spatially uncorrelated random process. For the propellant's

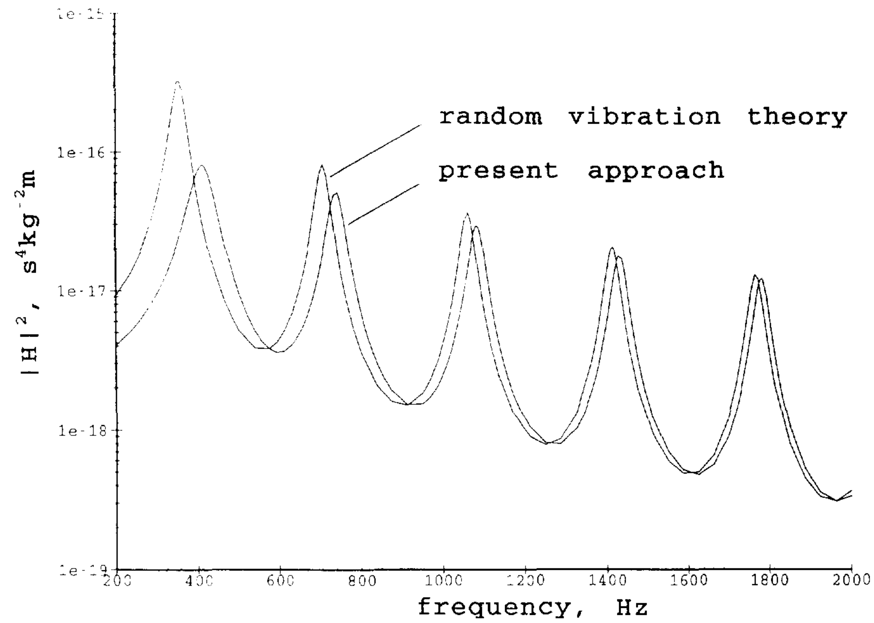

Fig. 2. Transfer function of the engine compartment vs frequency.

compartment the condition of constant mass per length unit is satisfied, hence eqn (30) is applicable. For numerical calculations the following parameters were taken: $\mu = M_1 = 100$ kg m^{-1}, $C_1 = 8\times10^7$ N, $C_2 = 5\times10^7$ N, $K = 5\times10^7$ N m^{-1}, $\kappa = 0.25$, $\xi_n = 250$ s^{-1}, $r-l = 1$ m, $L = 4$ m. Results of numerical calculations for the square of the absolute value of the transfer function $|H|^2 = S_{\langle u_2^2\rangle}S_f^{-1}$ are shown in Fig. 2. The upper curve corresponds to the absolutely isolated engine compartment, while the lower one was calculated by means of the present approach, i.e. eqn (30). The discordance is considerable for the first three resonances (notice that the vertical scale is logarithmic). Difference between the curves becomes negligible for $f > 1000$ Hz; hence, for this particular system the localization of vibration in engine compartment occurs at frequencies above 1000 Hz.

Since many assumptions and approximations have been made in the present approach, a comparative study to verify the accuracy of the approach is needed. To check the obtained theoretical results, a Finite Element analysis was performed. To this end, the first body (rocket, $0 \leqslant x \leqslant L$) was discretized by 50 degrees-of-freedom and the second one (engine compartment, $l \leqslant x \leqslant r$) by 20 degrees-of-freedom. The damping parameter κ of the first body is modelled by means of a complex stiffness matrix $C_1^* = C_1/(1-i\kappa)^2 = [(1-\kappa^2)/(1+\kappa^2)^2+i2\kappa/(1+\kappa^2)^2]C_1$, where C_1 is the stiffness matrix of the undamped discretized first body. The damping matrix of the second body is given as $D_2 = M_2\cdot\xi_n$, where M_2 is the mass matrix of the second discretized body. The averaged transfer function $|H(\omega)|^2$ has been determined by solving the complex matrix equation for the transfer function $\tilde{H}(\omega) = [K+i\omega D+\omega^2M]^{-1}\cdot\tilde{f}(\omega)$ and averaging $|\tilde{H}(\omega)|^2$ over the second body. The result of the computations is represented in Fig. 3 by the curve referred to as FE-computation. The discrepancy between the present approach and the FE analysis is observable in Fig. 3 at high frequencies and it is displayed by a frequency shift of the FE curves to high frequencies. This phenomenon is well-known and it is explained by the fact that the FE method is a modification of the Ritz method. The observed shift can be eliminated by increasing the number of degrees of freedom.

5.2. *Cover of an engine's head*

The vibrational design of engine covers is important for the prediction of engine noise. Let the cover be a box with all its five walls being thin plates of thickness h. In this case the box is a structural member having a constant mass per unit area $\mu = \rho h =$ const, where ρ is the mass density. Let $\mathbf{u}_c(\mathbf{r}, t)$ and $\mathbf{v}_n(\mathbf{r})$ denote the absolute displacement of the cover and

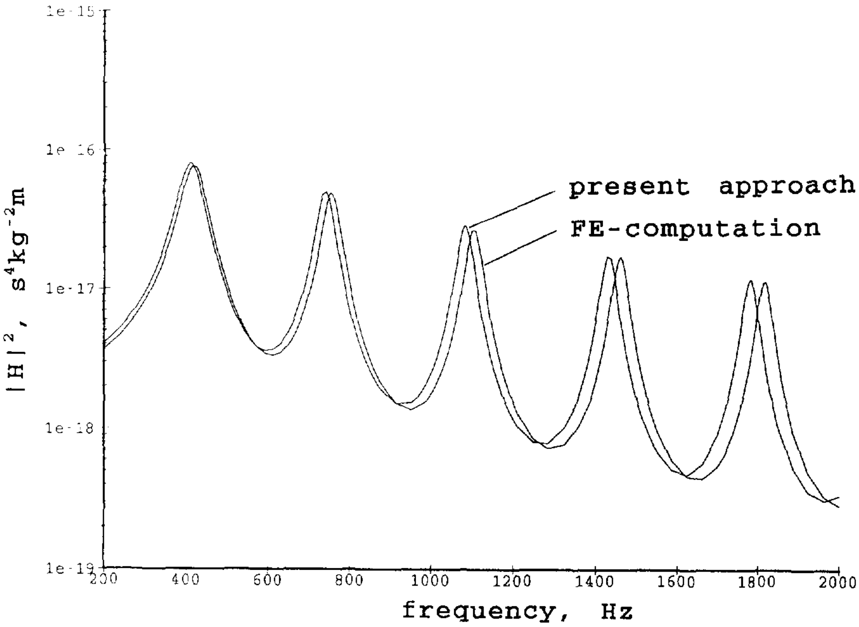

Fig. 3. Comparison of the present approach and FE-computation for the transfer function.

its normal modes, respectively. Provided that the normal modes $\mathbf{v}_n(\mathbf{r})$ are normalized, the generalized coordinates $q_n(t)$ are given as [cf. eqns (9) and (10)]

$$q_n(t) = \int_A m_2(\mathbf{r})\mathbf{v}_n(\mathbf{r}) \cdot \mathbf{u}_c(\mathbf{r}, t)\,\mathrm{d}A\,; \quad \int_A \mu(\mathbf{r})v_k(\mathbf{r})v_n(\mathbf{r})\,\mathrm{d}A = \delta_{kn}, \tag{33}$$

where $A = b_1b_2 + 2(b_1b_3 + b_2b_3)$ is the area of the box faces. An averaged value of the square of the velocity may be expressed in the form

$$\langle \dot{\mathbf{u}}_c^2 \rangle = \frac{1}{A}\int_A E\{\dot{\mathbf{u}}_c^2(\mathbf{r}, t)\}\,\mathrm{d}A = \frac{1}{\mu A}\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} E\{\dot{q}_j\dot{q}_n\}\int_A \mu\mathbf{v}_n(\mathbf{r}) \cdot \mathbf{v}_j(\mathbf{r})\,\mathrm{d}A = \frac{1}{\mu A}\sum_{n=1}^{\infty} E\{\dot{q}_n^2(t)\}. \tag{34}$$

Let us suppose that by analogy with Section 4 the influence of the engine and the gasket can be taken into account by the effective eigenfrequencies $\hat{\Omega}_n$ and the effective damping factor $\hat{\xi}_n$, i.e.

$$q_n(\omega) = \frac{1}{-\omega^2 + i\hat{\xi}_n\omega + \hat{\Omega}_n^2}\int_A f(\mathbf{r}, \omega) \cdot \mathbf{v}_n(\mathbf{r})\,\mathrm{d}A. \tag{35}$$

Now eqn (34) can be rewritten in the form

$$\langle \dot{\mathbf{u}}_c^2 \rangle = \frac{1}{\mu A}\sum_{n=1}^{\infty} E\{\dot{q}_n^2(t)\}$$

$$= \frac{1}{\mu A}\sum_{n=1}^{\infty}\int_A\int_A\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\mathbf{v}_n(\mathbf{r}) \cdot E\{\mathbf{f}(\mathbf{r}, \omega)\mathbf{f}(\mathbf{r}', \omega')\} \cdot \mathbf{v}_n(\mathbf{r}')\,\mathrm{e}^{i(\omega-\omega')t}\,\omega\omega'}{(-\omega^2 + i\hat{\xi}_n\omega + \hat{\Omega}_n^2)(-\omega'^2 - i\hat{\xi}_n\omega' + \hat{\Omega}_n^2)}\,\mathrm{d}A\,\mathrm{d}A'\,\mathrm{d}\omega\,\mathrm{d}\omega'. \tag{36}$$

The engine head and its cover form an "acoustical chamber". Since there is a little attenuation due to damping and the reflection of propagating waves is strong, the wave is reflected many times before it is absorbed. This results in a highly reverberant field, which is more or less uniformly distributed throughout the cover (Hodges and Woodhouse, 1986). Such a "rain-on-the-roof" loading (after Maidanik) is a typical spatial white noise, that can be expressed in the following form:

$$E\{\mathbf{f}(\mathbf{r},\omega)\mathbf{f}^*(\mathbf{r}',\omega')\} = S_p(\omega)\delta(\omega-\omega')\delta(\mathbf{r}-\mathbf{r}')\mathbf{I}. \tag{37}$$

where $S_p(\omega)$ is the spectral density of the acoustic wave pressure, $\mathbf{I}$ is a unit tensor and * denotes the complex conjugate of a quantity. Substitution of eqn (37) into (36) yields

$$\langle \dot{\mathbf{u}}_c^2 \rangle = \frac{1}{\mu A}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i(\omega-\omega')t}\,\omega\omega' S_p(\omega)\delta(\omega-\omega')\,d\omega\,d\omega'}{(-\omega^2+i\xi_n\omega+\hat{\Omega}_n^2)(-\omega'^2-i\xi_n\omega'+\hat{\Omega}_n^2)}$$

$$\times\int_A\int_A \mathbf{v}_n(\mathbf{r})\cdot\mathbf{I}\cdot\mathbf{v}_n(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')\,dA\,dA' = \frac{1}{\mu^2 A}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\frac{\omega^2 S_p(\omega)\,d\omega}{|-\omega^2+i\xi_n\omega+\hat{\Omega}_n^2|^2}. \tag{38}$$

For a lightly damped box and wide-band external driving, the integral in eqn (38) can be evaluated, to give (Bolotin, 1984)

$$\langle \dot{\mathbf{u}}_c^2 \rangle = \frac{\pi}{\mu^2 A}\sum_{n=1}^{\infty}\frac{S_p(\hat{\Omega}_n)}{\xi_n(\hat{\Omega}_n)}. \tag{39}$$

For the case of a high density of eigenfrequencies the sum in eqn (39) may be replaced by an integral (Bolotin, 1984)

$$\langle \dot{\mathbf{u}}_c^2 \rangle = \frac{\pi}{\mu^2 A}\int_0^{\infty}\frac{S_p(\Omega)}{\xi(\Omega)}\frac{dN}{d\Omega}\,d\Omega. \tag{40}$$

where the density of eigenfrequencies $dN/d\Omega$ is introduced. An asymptotic density of eigenfrequencies of the flexural vibrations of a plate $b_1 \times b_2$ is known to be independent of the boundary conditions and it is equal to

$$\frac{dN}{d\Omega} = \frac{b_1 b_2}{4\pi}\sqrt{\frac{\rho h}{D}}. \tag{41}$$

where D is the flexural rigidity of the plate [cf. Bolotin (1984)]. Non-degeneration of the dynamic fringe effect for plates admits an approximate matching of solutions for adjacent subregions. Consequently, the eigenfrequencies' density of the cover is equal to the sum of the densities of these plates, i.e.

$$\frac{dN}{d\Omega} = \frac{A}{4\pi}\sqrt{\frac{\rho h}{D}}. \tag{42}$$

Inserting this result into eqn (40) we have

$$\langle \dot{\mathbf{u}}_c^2 \rangle = \frac{1}{4\rho h\sqrt{\rho h D}}\int_0^{\infty}\frac{S_p(\Omega)\,d\Omega}{\xi(\Omega)}. \tag{43}$$

The spectral density of spatially averaged velocities of the cover is obtained from eqn (43)

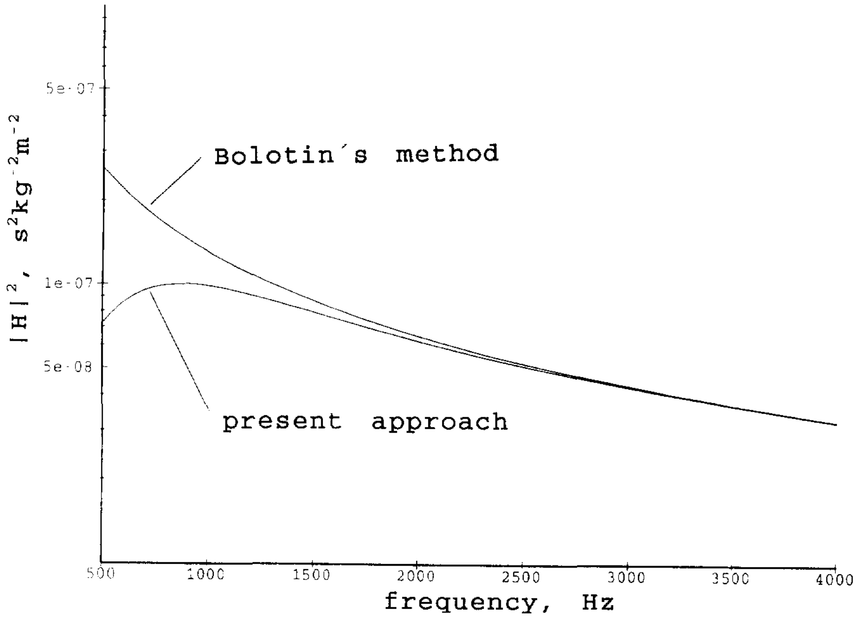

Fig. 4. Transfer function of the cover vs frequency.

$$S_{\langle u_c^2 \rangle}(\omega) = |H(\omega)|^2 S_p(\omega); \quad |H(\omega)|^2 = \frac{\pi}{4\rho h\sqrt{\rho h D}\hat{\xi}(\omega)}. \tag{44}$$

where H is the transfer function.

For numerical calculations the following parameters were taken: $b_1 = 0.2$ m, $b_2 = 0.45$ m, $b_3 = 0.08$ m, $h = 1\times10^{-3}$ m, $E = 2.1\times10^{11}$ N m^{-2}, $\rho = 7.8\times10^3$ kg m^{-3}, $\nu = 0.3$, $K = 0.9\times10^8$ N m^{-1}, $C_1 = 1.5\times10^9$ N, $M_1 = 250$ kg m^{-1}, $L = 1$ m. The critical ratio of damping is chosen to be 5×10^{-3} at all frequencies 5×10^2 Hz $\leqslant f \leqslant 4\times10^3$ Hz. We may neglect the frequency drift, eqn (31), since it does not affect the asymptotic density of eigenfrequencies. In Fig. 4 the curves of the square of the absolute value of transfer function $|H(\omega)|^2$ are shown. The upper curve corresponds to the absolutely isolated cover (Bolotin's method), while the lower one is calculated by means of the present approach, i.e. eqn (44). The curves actually coincide at high frequencies, thus the vibrations localize within the cover. In order to predict the engine noise, the engine born noise computations are usually performed for the frequency range from 500 to 3000 Hz [cf. Herster *et al.* (1994)]. The particular example considered shows that within this range the cover and the other part of the engine interact. Beyond this frequency range, the vibrations localize within the cover and, hence, for the frequencies above 3000 Hz this individual structural member may be considered separately, e.g. by means of the random vibration theory.

6. CONCLUDING REMARKS

Three methods, Bolotin's method of integral estimates, Statistical Energy Analysis and the theory of high-frequency vibration, were combined in order to forecast the vibrations of structural members of complex systems. This allows one to overcome certain shortcomings of these methods. The application of the theory of high-frequency vibration results only in certain integral fields of vibration, but a structural element itself is not represented in this theory. On the other hand, in the method of integral estimates a structural element is considered, but it is assumed to be absolutely isolated. The main result of the proposed concept is that any structural members may be considered separately. Other structural members and the coupling are taken into account by a frequency drift and an effective damping factor. Among these two parameters the effective damping factor is more important [cf. eqn (44)]. Under a wide-band driving a great many modes are excited, so that we can use an asymptotic density of eigenfrequencies which is not sensitive to a frequency drift.

Two practical problems were solved using this concept. Consideration of Figs 2 and 4 allows us to make some conclusions about the localization of vibrations, also known as strong vibration localization or normal mode localization (Hodges and Woodhouse, 1986; Ibrahim, 1987; Xie and Ariaratnam, 1994). Analysing Figs 2 and 4 one can specify the frequency domain where the results of the Bolotin method and of the present approach can be considered as coinciding with desired accuracy. It is also evident that the localization of vibrations actually takes place at high frequencies and the tendency to localization becomes stronger with the growth of frequency. This conclusion fully agrees with the central result of the papers by Pierre (1990) and Cha and Pierre (1991).

The phenomenon of the localization of high-frequency vibration is essential in the formation of the vibrational field of complex structures, which is rarely stated clearly in the structural dynamics literature. On the other hand, the strong localization is shown to be inherent at high frequencies and it changes drastically the dynamic properties of engineering structures. In all cases this means that the dynamical simulation should be performed with the localization of vibration being taken into account. The present study allows one to specify a frequency domain wherein a strong localization of vibration within structural members occurs.

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